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Remark on the solutions of the diffusion equation

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Abstract—There exists between the solutions of the diffusion equation and the solutions of the wave equation a relationship which enables new solutions of the diffusion equation to be obtained. Focus and splash mode solutions are discussed which are connected to the recently discovered mode solutions of the wave equation.

1. INTRODUCTION

THE DIFFUSION equation has been the subject of many works in different areas: heat conduction [1, 2], neutron scattering [3], wave mechanics [4] ... But a relationship between the solutions of the diffusion equation and the solutions of the wave equation passed largely unnoticed. This relationship is discussed here and it is proved that it leads to new solutions in unbounded media.

Using the heat equation as representative of the diffusion processes and to simplify the matter assume the diffusivity as unity so that the heat equation takes the form

$$\Delta\psi_2 - \partial_t\psi_2 = 0 \tag{1}$$

where Δ is the Laplacian operator and ∂_t the derivative with respect to time.

With the velocity of light unity, the wave equation is

$$\Delta\psi_1 - \partial_t^2\psi_1 = 0. \tag{2}$$

Then taking the Laplace transform of equations (1) and (2) with respect to time and using the well-known relation [5] between the inverse transforms $\mathscr{L}^{-1}(f(p))$ and $\mathscr{L}^{-1}(f(\sqrt{p}))$ where p is the symbolic variable one obtains the following relation between ψ_1 and ψ_2:

$$\psi_2(x,y,z,t) = \frac{1}{\sqrt{(\pi t)}}\int_0^\infty e^{-s^2/4t}\,\psi_1(x,y,z,s)\,ds \tag{3}$$

provided that $\partial_t\psi_1 = 0$ at $t = 0$ (in ref. [5], it is required wrongly that ψ_1 be also zero at $t = 0$).

For instance with

$$\psi_1(z,t) = \cos\alpha t\, e^{-i\alpha z} \tag{4a}$$

$$\psi_1(r,\phi,t) = e^{-in\phi}\cos\alpha t\,\tau_n(\alpha r) \tag{4b}$$

one obtains the elementary solutions

$$\psi_2(z,t) = e^{-i\alpha z}\,e^{-\alpha^2 t} \tag{5a}$$

$$\psi_2(r,\phi,t) = e^{-in\phi}\,e^{-\alpha^2 t}\,\tau_n(\alpha r) \tag{5b}$$

where α is an arbitrary scalar, r, ϕ the transverse coordinates, and τ_n denotes one of the usual Bessel functions of order n.

From now on relation (3) is used first to discuss the general plane, spherical, and cylindrical solutions of the heat equation in unbounded media and second to introduce new kinds of solutions.

2. GENERAL SYMMETRIC SOLUTIONS

2.1. *Plane solutions*

The general plane wave solution of the one-dimensional wave equation is

$$\psi_1(z,t) = \tfrac{1}{2}(F(z-t)+G(z+t)) \tag{6}$$

where F, G are arbitrary smooth functions while the factor 1/2 has been introduced for convenience. The condition $\partial_t\psi_1 = 0$ at $t = 0$ requires $F = G$. Then substituting

$$\psi_1(z,s) = \tfrac{1}{2}(F(z-s)+F(z+s)) \tag{6'}$$

into equation (3) gives the general plane solution of the one-dimensional diffusion equation in the form of a Wiener integral

$$\psi_2(z,t) = \frac{1}{2\sqrt{(\pi t)}}\int_{-\infty}^{\infty} e^{-s^2/4t}\,F(z-s)\,ds. \tag{7}$$

For instance, if $F(z-s)$ is the nth derivative $\delta^n(z-s)$ of the Dirac distribution, one obtains the instantaneous multiplet source solutions [1] of equation (1)

$$\psi_{2,n}(z,t) = \frac{(-1)^n}{2\sqrt{(\pi t)}}\left(\frac{\partial^n}{\partial s^n}e^{-s^2/4t}\right)_{s=z} \tag{8}$$

while $F(z-s) = e^{-i(z-s)}$ leads to equation (5a).

Relation (7) is usually obtained in a different way by using the Fourier transform and the convolution product [6]. In this case $F(z)$ is the initial distribution of temperature along $0z$.

2.2. *Spherical solutions*

First note that in spherical geometry the diffusion equation takes the form

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$$\frac{d^2}{dR^2}(R\psi_2)-\partial_t(R\psi_2)=0,\quad R^2=x^2+y^2+z^2 \qquad (9)$$

comparing equation (9) with the one-dimensional diffusion equation $\partial_z^2\psi_2-\partial_t\psi_2=0$ shows that one has just to substitute R and $R\psi_2$ to z and ψ_2 in equation (7) to obtain the spherical solutions of equation (1). This gives

$$\psi_2(R,t)=\frac{1}{2\sqrt{(\pi t)}}\int_{-\infty}^{\infty}e^{-s^2/4t}F(R-s)\frac{ds}{R}. \qquad (10)$$

For $F(R-s)=\delta^n(R-s)$, one obtains the instantaneous multiplet source solutions

$$\psi_{2,n}(R,t)=\frac{(-1)^n}{2\sqrt{(\pi t)}}\frac{1}{R}\left(\frac{\partial^n}{\partial s^n}e^{-s^2/4t}\right)_{s=R} \qquad (11)$$

which reduce for $n=1$ to the well-known result

$$\psi_{2,1}(R,t)=\frac{1}{4\sqrt{(\pi t^3)}}e^{-R^2/4t} \qquad (11')$$

corresponding to a point source.

2.3. *Cylindrical solutions*

For cylindrical solutions, the situation is not so simple since one has to cope with a two-dimensional problem. Wave equation (2) has the cylindrical solution [7]

$$\psi_1(r,t)=\frac{1}{2\pi}\int_0^{\infty}F(R-t)\frac{dz}{R},$$
$$R^2=x^2+y^2+z^2,\quad r^2=x^2+y^2 \qquad (12)$$

where F is a function such that its derivative tends to zero sufficiently fast at infinity.

The derivative of $\psi_1(r,t)+\psi_1(r,-t)$ is zero at $t=0$, so one may use equation (3). Then the general cylindrical solution of the diffusion equation takes the form

$$\psi_2(r,t)=\frac{1}{4\pi\sqrt{t}}\int_{-\infty}^{\infty}ds\,e^{-s^2/4t}\int_0^{\infty}F(R-s)\frac{dz}{R} \qquad (13)$$

provided that the second integral in equation (13) is smooth and bounded.

For instance for the harmonic solutions $F(R-s)=e^{i\alpha(R-s)}$ relation (13) leads to expression (5b) with $n=0$ and $\tau_0\equiv H_0^1$ where H_0^1 is the usual Hankel function. This last result comes from the relation [7]

$$\frac{1}{2\pi}\int_0^{\infty}e^{i\alpha(R-t)}\frac{dz}{R}=H_0^1(\alpha r)\,e^{-i\alpha t}. \qquad (14)$$

It does not seem that solutions (13) were previously known.

3. FOCUS AND SPLASH MODE SOLUTIONS

3.1. *Focus mode solutions*

Relation (3) is now used to obtain two new classes of solutions. One starts with the remark that the focus wave mode solutions of the hyperbolic partial differential equations have been introduced by Brittingham [8]. They correspond to waves propagating along $0z$ with a transverse Gaussian structure and a longitudinal Lorentzian structure. For the wave equation, they take the form [9]

$$\psi_{1,n}(\xi,\bar{\xi})=\frac{r^n}{(a-i\xi)^{n+1}}e^{-kr^2/(a-i\xi)}e^{-i(k\bar{\xi}+n\phi)} \qquad (15)$$

with $\xi=z-t$, $\bar{\xi}=z+t$. Here it is assumed that the parameters a,k,n are arbitrary but to obtain bounded continuous solutions of equation (2), one must take a,k real with $ak>0$ and $n=0,1,2\ldots$

Now $\psi_{1,n}(\bar{\xi},\xi)$ obtained by interchanging ξ and $\bar{\xi}$ is also a solution so that $\psi_{1,n}(\xi,\bar{\xi})+\psi_{1,n}(\bar{\xi},\xi)$ has its time derivative null at $t=0$ in agreement with the condition required to use equation (3). Taking this result into account together with equation (15), one obtains from equation (3)

$$\psi_{2,n}(\mathbf{r},t;k)=\frac{r^n}{\sqrt{(\pi t)}}e^{-i(n\phi+kz)}\int_{-\infty}^{\infty}\frac{ds}{(a-iz+is)^{n+1}}$$
$$\times\exp\left(-\frac{s^2}{4t}-iks-\frac{kr^2}{a-iz+is}\right) \qquad (16)$$

with $\mathbf{r}=(r,\phi,z)$.

Relation (16) defines the focus mode solutions of the diffusion equation provided the integral exists. But unlike focus modes (15) the structure of these solutions is not easy to visualize. One may write equation (16) in the form

$$\psi_{2,n}(\mathbf{r},t;k)=e^{-ikz-k^2t}\,\Phi_n(\mathbf{r},t;k) \qquad (17)$$

where Φ_n is the structure function

$$\Phi_n(\mathbf{r},t;k)=\frac{r^n e^{-in\phi}}{\sqrt{(\pi t)}}\int_{-\infty}^{\infty}\frac{ds}{(a-iz+is)^{n+1}}$$
$$\times e^{-(s/2\sqrt{t}+ik\sqrt{t})^2}e^{-kr^2/(a-iz+is)} \qquad (17')$$

$\mathbf{\Phi}_n$ is single valued for integer n and bounded in the transverse direction for $n\geqslant 0$ and $k\,Re\,a>0$ (assuming k real).

Using the saddle point method of integration [6] leads to the following approximation of Φ_n:

$$\Phi(\mathbf{r},t;k)\cong\frac{r^n e^{-in\phi}}{(a-iz+2kt)^{n+1}}e^{-kr^2/(a-iz+2kt)} \qquad (18)$$

which is also a Gaussian transversally and Lorentzian longitudinally.

3.2. *Splash mode solutions*

Returning to equation (2), in most of the physical processes the wave energy density is proportional to $|\psi_1|^2$. Then it is easy to prove that the total energy

$$\int_{-\infty}^{\infty}d\xi\int_0^{\infty}r\,dr\int_0^{2\pi}d\phi|\psi_{1,n}|^2$$

in the focus wave modes (15) is infinite [9, 10]. It is not a drawback *per se*. After all the plane wave solutions also share this property. Nevertheless it is interesting to find solutions with finite energy. Ziolkowski [11] proved that the splash wave mode solutions obtained as a weighted superposition of the focus wave modes have this property for a correct choice of the weight function.

Similarly assuming k real, one can define the splash mode solutions $\hat{\psi}_{2,n}$ of equation (1) as a weighted mean of solutions (16)

$$\hat{\psi}_{2,n}(\mathbf{r},t) = \int_{-\infty}^{\infty} \mathrm{d}k f(k)\psi_{2,n}(\mathbf{r},t;k) \quad (19)$$

where $f(k)$ is a convenient weight function. Using relation (16) with $a = -ib$ one obtains

$$\hat{\psi}_{2,n}(\mathbf{r},t) = \frac{r^n \mathrm{e}^{-in\phi}}{\sqrt{(\pi t)}} \int_{-\infty}^{\infty} \frac{\mathrm{d}s\,\mathrm{e}^{-s^2/4t}}{(b+z-s)^{n+1}} \times \int_{-\infty}^{\infty} \mathrm{d}k f(k)\,\mathrm{e}^{-ik\theta/s} \quad (20)$$

with

$$\theta(s) = z+s+\frac{r^2}{b+z-s}. \quad (20')$$

As a first example, $f(k) = 1/k$ then equation (20) becomes

$$\hat{\psi}_{2,n}(\mathbf{r},t) = \frac{r^n \mathrm{e}^{-in\phi}}{\sqrt{(\pi t)}} \int_{-\infty}^{\infty} \frac{\mathrm{d}s\,\mathrm{e}^{-s^2/4t}}{(b+z-s)^{n+1}} H(\theta(s)) \quad (21)$$

where H denotes the Heaviside function.

Let s_1, s_2, be the roots of the equation $\theta(s) = 0$. According to equation (20′) one obtains

$$s_{1,2} = \tfrac{1}{2}(b \pm \sqrt{((b+2z)^2+4r^2)}) \quad (22)$$

with b real these roots are real and s_1 being the smaller root one has

$$H(\theta(s)) = H(s-s_1)-H(s-s_2). \quad (23)$$

Substituting equation (23) into equation (21) gives

$$\hat{\psi}_{2,n}(\mathbf{r},t) = \frac{r^n \mathrm{e}^{-in\phi}}{\sqrt{(\pi t)}} \int_{s_1}^{s_2} \frac{\mathrm{d}s\,\mathrm{e}^{-s^2/4t}}{(b+z-s)^{n+1}}. \quad (24)$$

Since $b+z$ is inside the interval (s_1, s_2) one must be careful in defining equation (24) as a Cauchy principal value around $s = b+z$.

As a second example assume $f(k) = 1$ so that equation (20) becomes

$$\hat{\psi}_{2,n}(\mathbf{r},t) = \frac{r^n \mathrm{e}^{-in\phi}}{\sqrt{(\pi t)}} \int_{-\infty}^{\infty} \mathrm{d}s\,\delta(\theta(s)) \frac{\mathrm{e}^{-s^2/4t}}{(b+z-s)^{n+1}} \quad (25)$$

where δ is the Dirac distribution. Using the well-known relation [5]

$$\delta(\theta(s)) = \Sigma \frac{\delta(s-s_n)}{|\theta'(s_n)|}$$

in which the summation has to be extended over all the points s_n where $\theta(s)$ changes sign, one obtains

$$\hat{\psi}_{2,n}(\mathbf{r},t) = \frac{r^n \mathrm{e}^{-in\phi}}{\sqrt{(\pi t)}} \left(\frac{\mathrm{e}^{-s_1^2/4t}}{(z+s_2)^{n+1}\left(1+\frac{r^2}{(z+s_2)^2}\right)} + \frac{\mathrm{e}^{-s_2^2/4t}}{(z+s_1)^{n+1}\left(1+\frac{r^2}{(z+s_1)^2}\right)} \right) \quad (26)$$

with s_1, s_2 still given by equation (22).

For $b = 0$ one has $s_2 = -s_1 = \sqrt{(z^2+r^2)}$ and equation (26) becomes

$$\hat{\psi}_{2,n}(\mathbf{r},t) = \frac{r^n \mathrm{e}^{-in\phi}}{2\sqrt{(\pi t(z^2+r^2))}} \mathrm{e}^{-(r^2+z^2)/4t} \times \left(\frac{1}{(z+\sqrt{(z^2+r^2)})^n} - \frac{1}{(z-\sqrt{(z^2+r^2)})^n} \right) \quad (27)$$

which gives for $n = 1$

$$\hat{\psi}_{2,1}(\mathbf{r},t) = \frac{\mathrm{e}^{-i\phi}}{r\sqrt{(\pi t)}} \mathrm{e}^{-(r^2+z^2)/4t} \quad (27')$$

a solution which is not bounded for $r \to 0$.

3.3. *Generalization*

It is now proved that the two classes of the previous solutions can be generalized. Note that u the set of the variables $(r, \phi, \bar{\xi})$ and consider the transformation

$$\psi_{1,g}(u,\xi) = \int_{-\infty}^{\infty} g(\xi-s)\psi_1(u,s)\,\mathrm{d}s \quad (28)$$

where g is a differentiable function null at infinity and such that integral (28) exists.

It is easy to show that if ψ_1 is a solution of equation (2), $\psi_{1,g}$ is also a solution of equation (2). Then substituting equation (15) into equation (28) supplies the generalized focus wave mode solutions $\psi_{1,g;n}$ and substituting $\psi_{1,g;n}$ into equation (3) gives the generalized focus mode solutions $\psi_{2,g;n}$ of equation (1).

A particular attractive case happens when g is the following Gaussian function shifted in direct and Fourier transform spaces:

$$g(s,\xi;w) = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left(\frac{-(s-\xi)^2}{4\sigma^2} + iw(s-\xi) \right) \quad (29)$$

where σ, w, are positive scalars. Then equation (28) defines the Gabor transformation [12] of ψ_1.

Using $\psi_{2,g;n}$ instead of $\psi_{2,n}$ in equation (19) leads to the generalized splash mode solutions $\hat{\psi}_{2,g;n}$. It can be remarked that with equation (29), one should obtain another class of splash mode solutions by weighting with respect to the scalar w. This will not be elaborated on further.

4. CONCLUSIONS

It is curious that relation (3) went unnoticed (ref. [5] is an exception) since many authors used the Laplace transform to solve both equations (1) and (2) and since the relation between $\mathscr{L}^{-1}(f(p))$ and $\mathscr{L}^{-1}(f(\sqrt{p}))$ is well known. Perhaps the reason is that relation (3) seems to be not very useful for boundary value problems. For initial value problems one obtains from relation (3), in addition to the usual solutions, new classes of solutions such as the focus and splash modes the properties of which as well as the physical meaning have still to be investigated.

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REMARQUES SUR LES SOLUTIONS DE L'EQUATION DE DIFFUSION

Résumé—Entre les solutions de l'équation de diffusion et celles de l'équation d'onde, il existe une relation qui permet d'obtenir de nouvelles solutions de l'équation de diffusion. On discute les solutions de type foyer et de type splash en connection avec des solutions récentes de l'équation d'onde.

ANMERKUNG ZUR LÖSUNG DER DIFFUSIONSGLEICHUNG

Zusammenfassung—Zwischen der Lösung der Diffusionsgleichung und der Lösung der Wellengleichung besteht ein Zusammenhang, welcher zu neuen Lösungsätzen der Diffusionsgleichung führt. Es werden die "Focus-Methode" und die "Splash-Methode" diskutiert, welche mit den vor kurzem gefunden Lösungsmethoden der Wellengleichung zusammenhängen.

О РЕШЕНИЯХ УРАВНЕНИЯ ДИФФУЗИИ

Аннотация—Между решениями уравнения диффузии и волнового уравнения существует зависимость, которая дает возможность получить новые решения уравнения диффузии. Обсуждаются решения для фокуса и вспышки, которые связаны с недавно открытыми решениями волнового уравнения.